Lecture 5. Random Signal Analysis

- Random Variables and Random Processes
- Signal Transmission through a Linear System

Discrete Random Variables

- A discrete random variable takes on a countable number of possible values.

Suppose that a discrete random variable X takes on one of the values $x_1, \ldots, x_n$.

✓ Distribution functions:

$$\sum_{i=1}^{n} p(x_i) = 1$$

Probability Mass Function: $p(x_i) = \Pr\{X = x_i\}$

Cumulative Distribution Function: $F(a) = \Pr\{X \le a\} = \sum_{x_i \le a} p(x_i)$

✓ Moments:

Expected Value, or Mean: $\mu_X = E[X] = \sum_{i=1}^{n} x_i p(x_i)$

The m-th Moment: $E[X^m] = \sum_{i=1}^{n} x_i^m p(x_i), \; m = 1, 2, \ldots$

Continuous Random Variables

- A continuous random variable has an uncountable set of possible values.

X is a continuous random variable if there exists a nonnegative function f, defined for all real $x \in (-\infty, \infty)$, having the property that for any set B of real numbers, $\Pr\{X \in B\} = \int_B f(x)dx.$

- ✓ f is called the **probability density function (pdf)** of X, denoted as: $f_X(x)$

$$\int_{-\infty}^{\infty} f_X(x)dx = 1$$

- ✓ **Cumulative Distribution Function (cdf):** $F_X(a) = \int_{-\infty}^{a} f_X(x)dx$
- ✓ **Expected Value, or Mean:** $\mu_X = E[X] = \int_{-\infty}^{\infty} x f_X(x)dx$
- ✓ **The m-the Moment:** $E[X^m] = \int_{-\infty}^{\infty} x^m f_X(x)dx, \;\; m = 1, 2, \ldots$

Variance

- Define *variance* of a random variable X as:

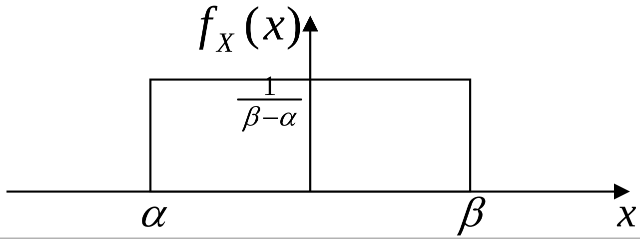
$$\mathrm{Var}[X] = E\left[(X - E[X])^2\right]$$

  - ✓ $\mathrm{Var}[X]$ describes how far apart X is from its mean on the average.
  - ✓ $\mathrm{Var}[X]$ can be also obtained as: $\mathrm{Var}[X] = E[X^2] - \left(E[X]\right)^2$
  - ✓ $\mathrm{Var}[X]$ is usually denoted as σ_X^2 .
  - ✓ The square root of Var[X], σ_X , is called the **standard deviation** of X.

Example 1. Uniform Distribution

- X is a uniform random variable on the interval (α, β) if its pdf is given by

$$f_X(x) = \begin{cases} \dfrac{1}{\beta - \alpha} & \alpha \le x \le \beta \\ 0 & otherwise \end{cases}$$



✓ cdf:

$$F_X(a) = \int_{-\infty}^{a} f_X(x)dx = \begin{cases} 0 & a \le \alpha \\ \dfrac{a-\alpha}{\beta-\alpha} & \alpha < a < \beta \\ 1 & a \ge \beta \end{cases}$$

✓ Mean: $\mu_X = \int_{-\infty}^{\infty} x f_X(x)dx = \dfrac{\beta + \alpha}{2}$

✓ The second moment:

$$E[X^2] = \int_{-\infty}^{\infty} x^2 f_X(x)dx = \frac{\beta^2 + \alpha\beta + \alpha^2}{3}$$

✓ Variance:

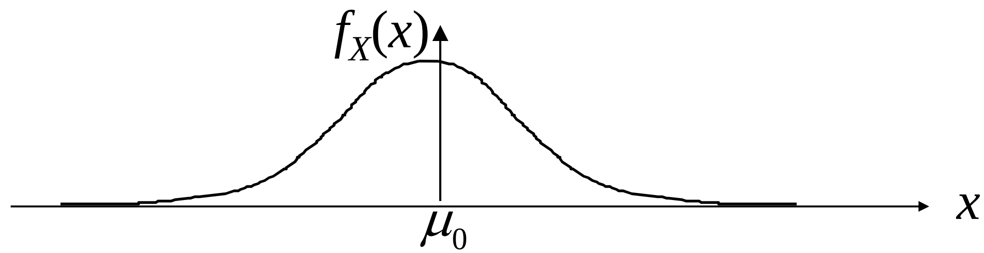
$$\sigma_X^2 = E[X^2] - \mu_X^2$$

$$= \frac{(\beta - \alpha)^2}{12}$$

Example 2. Gaussian (Normal) Distribution

- X is a Gaussian random variable with parameters μ_0 and σ_0^2 if its pdf is given by

$$f_X(x) = \frac{1}{\sqrt{2\pi}\sigma_0} \exp\left(-\frac{(x-\mu_0)^2}{2\sigma_0^2}\right)$$



- X is denoted as $X \sim \mathcal{N}(\mu_0, \sigma_0^2)$

✓ Mean: $\mu_X = \mu_0$

✓ Variance: $\sigma_X^2 = \sigma_0^2$

✓ cdf: $F_X(a) = \int_{-\infty}^{a} f_X(x)dx = 1 - \int_{a}^{\infty} \frac{1}{\sqrt{2\pi}\sigma_0} \exp\left\{-\frac{(x-\mu_0)^2}{2\sigma_0^2}\right\} dx$

$$\overset{z=\frac{x-\mu_0}{\sigma_0}}{=} 1 - \int_{\frac{a-\mu_0}{\sigma_0}}^{\infty} \frac{1}{\sqrt{2\pi}} \exp\left\{-\frac{z^2}{2}\right\} dz = 1 - Q\left(\frac{a-\mu}{\sigma_0}\right)$$

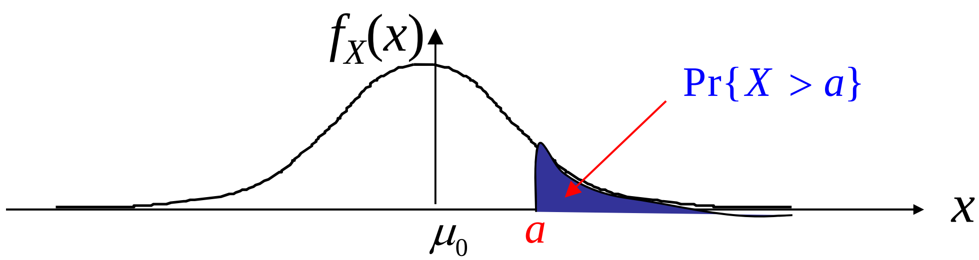
$$Q(\alpha) \triangleq \int_{\alpha}^{\infty} \frac{1}{\sqrt{2\pi}} \exp\left\{-\frac{x^2}{2}\right\} dx$$

More about Q Function

$$Q(\alpha)=\int_{\alpha}^{\infty}\frac{1}{\sqrt{2\pi}}\exp\left\{-\frac{x^2}{2}\right\}dx$$

- $Q(\alpha)$ is a decreasing function of α.
- For $X\sim\mathcal{N}(\mu_0,\sigma_0^2)$,

$$\Pr\{X>a\}=\int_a^{\infty}f_X(x)dx=\int_a^{\infty}\frac{1}{\sqrt{2\pi}\sigma_0}\exp\left\{-\frac{(x-\mu_0)^2}{2\sigma_0^2}\right\}dx=Q\left(\frac{a-\mu_0}{\sigma_0}\right)$$



Read the textbook Sec. 4.1.4 for more discussion about Q function.

Random Processes

- Sample values of a random process at time t_1, t_2, …, are a collection of random variables $\{X(t_1), X(t_2), \ldots\}$.
  - Continuous-time random process: $t \in \mathbb{R}$ (set of real numbers)
  - Discrete-time random process: $t \in \mathbb{Z}$ (set of integers)

- Statistical description of random process X(t)
  - A complete statistical description of a random process $X(t)$ is known if for any integer n and any choice of $(t_1,\ldots,t_n) \in \mathbb{R}^n$, the joint PDF of $(X(t_1),\ldots,X(t_n))$ is given.

Difficult to be obtained!

Statistical Averages

- The mean of the random process X(t):

$$\mu_X(t_k) = E[X(t_k)] = \int_{-\infty}^{\infty} x f_{X(t_k)}(x)dx$$

$X(t_k)$ is the random variable obtained by observing the random process $X(t)$ at time t_k, with the pdf $f_{X(t_k)}(x)$.

- The autocorrelation function of the random process X(t):

$$R_X(t_1,t_2) = E[X(t_1)X(t_2)] = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} x_1 x_2 f_{X(t_1),X(t_2)}(x_1,x_2)dx_1dx_2$$

$f_{X(t_1),X(t_2)}(x_1,x_2)$ is the joint pdf of $X(t_1)$ and $X(t_2)$.

Power and Power Spectrum of Random Signal

Time Domain

Deterministic signal s(t): $P_s = \lim_{T\to\infty} \frac{1}{T}\int_{-T/2}^{T/2} |s(t)|^2 \, dt$

Random signal (described as a random process X(t)):

$$P_X = E\left[\lim_{T\to\infty} \frac{1}{T}\int_{-T/2}^{T/2} X^2(t)dt\right] = \lim_{T\to\infty} \frac{1}{T}\int_{-T/2}^{T/2} R_X(t,t)dt$$

Frequency Domain

Deterministic signal:

$$P_s = \int_{-\infty}^{\infty} G_s(f)df, \quad G_s(f) \triangleq \lim_{T\to\infty} \frac{1}{T}|S_T(f)|^2 \Leftrightarrow \lim_{T\to\infty} \frac{1}{T}\int_{-T/2}^{T/2} s(t+\tau)s^*(t)dt$$

Random signal:

$$P_X = \int_{-\infty}^{\infty} G_X(f)df, \quad G_X(f) \triangleq \lim_{T\to\infty} \frac{1}{T} E|X_T(f)|^2 \Leftrightarrow \lim_{T\to\infty} \frac{1}{T}\int_{-T/2}^{T/2} R_X(t+\tau,t)dt$$

Example 1. Wide-Sense Stationary (WSS) Processes

- A random process $X(t)$ is wide-sense stationary (WSS) if the following conditions are satisfied:
  - $\mu_X(t) = E[X(t)]$ is independent of t;
  - $R_X(t_1, t_2)$ depends only on the time difference $\tau = t_1 - t_2$, and not on t_1 and t_2 individually.

✓ Power: $P_X = \lim_{T\to\infty} \frac{1}{T}\int_{-T/2}^{T/2} R_X(t,t)dt = R_X(0)$

✓ Power spectrum: $G_X(f) \Leftrightarrow \lim_{T\to\infty} \frac{1}{T}\int_{-T/2}^{T/2} R_X(t+\tau,t)dt = R_X(\tau)$

Example 2. Cyclostationary Processes

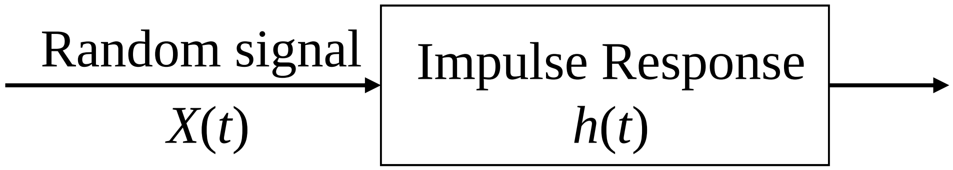
- A random process $X(t)$ with mean $\mu_X(t)$ and autocorrelation function $R_X(t+\tau, t)$ is called **cyclostationary**, if both the mean and the autocorrelation are periodic in t with some period T_0, i.e., if
  - $\mu_X(t+T_0) = \mu_X(t)$
  - $R_X(t+\tau+T_0, t+T_0) = R_X(t+\tau, t)$

✓ Power: $P_X = \lim_{T\to\infty} \frac{1}{T}\int_{-T/2}^{T/2} R_X(t,t)dt = \frac{1}{T_0}\int_{-T_0/2}^{T_0/2} R_X(t,t)dt$

✓ Power spectrum: $G_X(f) \Leftrightarrow \lim_{T\to\infty} \frac{1}{T}\int_{-T/2}^{T/2} R_X(t+\tau,t)dt = \frac{1}{T_0}\int_{-T_0/2}^{T_0/2} R_X(t+\tau,t)dt$

Signal Transmission through a Linear System

Linear Time Invariant (LTI) System



$$Y(t) = X(t) * h(t) = \int_{-\infty}^{\infty} X(\tau)h(t-\tau)d\tau$$

- If a WSS random process $X(t)$ passes through an LTI system with impulse response $h(t)$, the output process $Y(t)$ will be also WSS with mean

$$\mu_Y = \mu_X \int_{-\infty}^{\infty} h(t)dt = \mu_X H(0)$$

autocorrelation

$$R_Y(\tau) = R_X(\tau) * h(\tau) * h(-\tau)$$

and power spectrum

$$G_Y(f) = G_X(f)|H(f)|^2$$

Example 3. Gaussian Processes

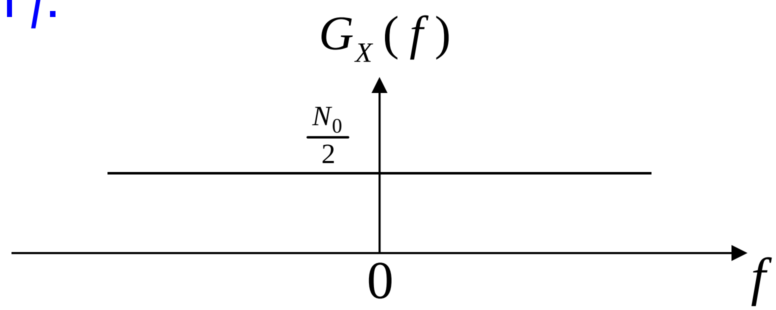
- A random process $X(t)$ is a **Gaussian** process if for all n and all $(t_1, \ldots, t_n)$, the random variables $\{X(t_i)\}_{i=1}^{n}$ have a jointly Gaussian pdf.

✓ For Gaussian processes, knowledge of the mean and auto-correlation gives a complete statistical description of the process.

✓ If a Gaussian process $X(t)$ is passed through an LTI system, the output process $Y(t)$ will also be a Gaussian process.

Example 4. White Processes

- A random process $X(t)$ is called a **white** process if it has a flat spectral density, i.e., if $G_X(f)$ is a constant for all f.



✓ Power: $P_X = \int_{-\infty}^{\infty} G_X(f)df = \infty$

✓ Autocorrelation:

$$G_X(f) \Leftrightarrow R_X(\tau) = \frac{N_0}{2}\delta(\tau)$$

$\frac{N_0}{2}$: two-sided power spectral density

✓ If a white process $X(t)$ passes through an LTI system with impulse response $h(t)$, the output process $Y(t)$ will not be white any more.

Power spectrum of $Y(t)$: $G_Y(f) = \frac{N_0}{2}|H(f)|^2$

Power of $Y(t)$: $P_Y = \frac{N_0}{2}\int_{-\infty}^{\infty}|H(f)|^2 df = \frac{N_0}{2}\int_{-\infty}^{\infty} h^2(t)dt$